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ABSTRACT

The focus in this report is the concept of what it means to "do mathematics" in the context of a group working committee of teachers and researchers committed to reforming mathematics teaching. During a regular meeting of this group of teachers and researchers, the participants explored the mathematical reasoning embedded in one fourth-grade student's unusual solution to a routine multiplication problem. The analysis prompted a framework for considering in a teacher's group what it means to do mathematics. It is argued that these participants conducted three types of investigation into the reasoning used by the student: psychological, pedagogical, and propositional. By focusing their exploration within the student's sense-making, the members situated their learning of the practice of mathematics teaching and of the subject matter of mathematics within their own practice of teaching. It is suggested that "doing mathematics" for teachers requires more than just additional college mathematics courses; doing mathematics also requires learning how to make sense of someone's reasoning, however strange it may seem at first glance. This can only be learned by actually "doing mathematics" in a classroom setting and discussing the process with others. (Contains 18 references.) (NAV)

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[1]Formerly known as the National Center for Research on Teacher Education (1985-1990), the Center was renamed in 1991.

Abstract

We examine what it might mean to *do mathematics* in the context of a teacher group committed to reforming mathematics teaching. To this end, we analyze one meeting of a group of teachers and researchers who meet regularly to investigate their own mathematics teaching. During this meeting, the group explored the mathematical reasoning embedded in one student's unusual solution to a routine multiplication problem. Our analysis leads us to develop a framework for considering what it means to do mathematics in a teachers' group. We argue that participants conducted three types of investigation: psychological, pedagogical, and propositional. By focusing their exploration within the student's sense-making—an integral part of teachers' daily practice—the members situated their learning of the practice of mathematics teaching and of the subject matter of mathematics within their own practice of teaching.

"He Knows There's Six 100s in 26?" An Investigation Into What It Means to *Do Mathematics* in a Teacher Group

Stephen P. Smith and Helen Featherstone

The National Council of Teachers of Mathematics (NCTM 1989, 1991) has argued that teachers should engage students in "doing mathematics": investigating and communicating mathematical ideas within a learning community of peers. However, few teachers have had learning experiences of this sort or have models on which to draw as they attempt to meet the challenge of creating such classroom environments. Reformers and educators committed to enacting this vision of mathematics education recognize the challenge in supporting teachers' efforts to learn to teach in these new ways:

> The kind of teaching envisioned in these standards is significantly different from what many teachers themselves have experienced as students in mathematics classes. Because teachers need time to learn and develop this kind of teaching practice, appropriate and ongoing professional development is crucial. Good instructional and assessment materials and the latitude to use them flexibly are also keys to the process of change . . . We cannot expect teachers to respond simultaneously to several different calls for change or other new demands. Change is difficult and will take time and reliable, systematic support. (NCTM 1991, pp. 2-3)

The support that the *Standards*[2] refers to may need to come through a variety of channels. Whatever the sources, this support should facilitate teachers developing their understanding of how children learn; of representations, tools, and activities which facilitate learning; and of the subject matter itself.

[2]*Standards* refers to *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and *Professional Standards for Teaching Mathematics* (NCTM 1991).

WHAT DO TEACHERS NEED TO KNOW?
HOW CAN THEY LEARN IT?

Various mathematics educators (e.g., Ball 1989; Ball and McDiarmid 1990; Even and Lappan 1994; Schifter 1993; Thompson et al. 1994) have argued that one of the challenges in reforming teacher education is for teachers to develop their subject-matter knowledge. Reformers argue that teachers need to have both a deeper and a different understanding of mathematics. Nevertheless, as Schifter and Bastable (1995) have recently pointed out, much of the research published to date focuses on changing teachers' "epistemological and pedagogical conceptions" rather than on the development of their *mathematical* understanding (Knapp and Peterson 1995; Russell and Corwin 1993; Wood, Cobb, and Yackel 1991). They go on to raise questions about the kinds of mathematical understandings teachers need and how to help teachers learn about "mathematics in terms of underlying conceptual issues" (Schifter and Bastable 1995, p. 1).

We share Schifter and Bastable's concern for the development of these mathematical understandings. Teachers need these understandings for a variety of reasons. One that we find particularly compelling is that they help adults to make sense of children's explanations of their reasoning. Teachers need rich understanding of mathematics to "see" the connections and constructions students are making. In classrooms where students are presenting their solutions, questioning each other, and making mathematical arguments, the ability of the members of the learning community—teachers *and* students—to make sense of children's reasoning seems crucial. Our recent experiences have led us to conclude that the challenge of making sense of students' reasoning can be both a reason for and a pathway to teachers expanding their own understanding of mathematics.

We work with a teacher group where the teachers shape each meeting by bringing issues from their daily practice.[3] Our conversations help us think about the parallels between the learning communities teachers who have been inspired by the *Standards* try

[3]For other examples of teachers' groups, see Wilcox et al. 1991; Schifter and Fosnot 1993; Duckworth 1987.

to create in their classrooms and those they seek to create for themselves as students of this teaching. We are learning that in order to acquire the knowledge and skills they need to create the learning communities they envision, teachers seek out ways to engage in *doing mathematics* that parallel the integrated investigations they create with their own students.

In the next section, we explore what it can mean to do mathematics in a teacher group: What does it look like? More specifically, if we explicitly broaden our focus, what do we see occurring as this group of teachers and researchers explores problems from their practices? Later, we will explore parallels between the ways children pursue investigations when doing math in classrooms that take the *Standards* seriously and the ways members of the Investigating Mathematics Teaching (IMT) group pursue investigations when working on mathematics arising out of their own problems of practice.

OUR CONTEXT: THE IMT GROUP

Seven teachers and three researchers of the National Center for Research on Teacher Learning (NCRTL) started meeting weekly in the fall of 1991, viewing and discussing video tapes of a third-grade classroom where students were working on integers.[4] Although the group originally convened as a one-term, master's-level course, members have continued in a voluntary association of teachers and university researchers who are committed to thinking hard about reform-oriented ways of teaching math and to addressing problems of practice. Over the ensuing three years, membership has ranged from seven to nine teachers, with six of the original teachers and the three researchers remaining part of the group. Members have left the group for various reasons, and new members have been added as participants' colleagues have heard about the group and

[4]These materials were generated by Mathematics and Teaching Through Hypermedia (M.A.T.H.). Funded by the National Science Foundation, the M.A.T.H. project documented the teaching and learning in the classrooms of Deborah Ball and Magdalene Lampert. Over the course of the 1989-1990 academic year, Ball, Lampert, and colleagues videotaped many mathematics lessons as well as interviews with students, mathematicians, and mathematics educators. They saved and reproduced all student work including homework, tests, and the students' math journals. A team of graduate students kept field notes on the mathematical and pedagogical issues raised in each lesson. Ball and Lampert kept teaching journals in which they daily recorded their reflections on lessons.

expressed interest in joining. Over the first two years, the members gradually changed their focus from the classroom of a nonmember to the classrooms of the participant teachers. This change of focus occurred gradually; teachers moved from describing problems of practice to accepting our invitation to videotape in their classrooms and to bring segments of these tapes into IMT meetings.[5]

In the first term, when discussions usually focused on a videotape of a third-grade classroom, the researchers sometimes introduced sessions by asking members to work on mathematics problems taken directly from the math class we planned to discuss.[6] After this first term, teachers increasingly defined the agenda. Because the members of the IMT group "teach against the grain" (Cochran-Smith 1991), they have looked for support in the group. The teachers have sought support on issues that cut across the practice of teaching: developing learning communities, managing relations with colleagues, balancing breadth and depth, designing mathematical tasks, and guiding students' exploration of the mathematical territory. Issues involving mathematical content have arisen when individual members sought help from the group with pedagogical problems. When such issues have become focal, the group explored the mathematics in different ways and to different extents.

OUR ANALYSIS

The ideas which we present in this paper emerged from our analysis of transcripts of a number of IMT meetings. We were curious about the parallels between the discourse within the group and that within the classrooms of the teachers. To pursue this, we examined meetings in which the doing of mathematics seemed to play a prominent role. The theoretical framework which we present here grew out of our conversations around these meetings. The IMT meeting that we excerpt here occurred as we were developing the framework. Concurrently, the members were developing projects which focused on teaching and learning of difficult mathematics.

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[5]We have audiotaped all IMT meetings, audiotaped regular interviews of the members, and videotaped their classes upon request. All interviews and many meetings have been transcribed.

[6]Examples: "Write number sentences equal to 10" and "Find a way to represent $-\frac{3}{} - (-\frac{2}{})$."

In this paper, we present our analysis of one occasion when the IMT group consciously and seriously pursued the mathematics which arose out of one member's problem of practice. In the group's conversation around the mathematics of one student's unusual solution to a rather straightforward fourth-grade problem, we see a parallel with the types of interactions that occur in classrooms of teachers implementing a vision of mathematics education based on the NCTM *Standards*; in both communities, we see participants conducting what we have come to term *psychological*, *pedagogical*, and *propositional* investigations.

DOING MATHEMATICS

We believe that trying to teach mathematics for understanding entails the following challenges: exploring and developing one's own understanding of mathematics, exploring and developing one's understanding of how children make sense of mathematics, and finding ways to help one's students connect with and make sense of mathematical ideas. As we analyze our data, we have developed the conjecture that members create opportunities within the IMT group to explore and develop these understandings through various investigations.

In the IMT group, *doing mathematics* emerges through teachers investigating their own mathematics teaching.[7] When *doing mathematics*, members conduct three types of investigations: pedagogical, psychological, and propositional. In this context, the members of the group construct pedagogical, psychological, and propositional knowledge as they inquire into mathematical situations which they bring from their classrooms. Their investigations enable them to construct knowledge: how to structure learning opportunities, how children think and reason, and how mathematics works. The following figure represents the authors' current thinking about the relationship among the investigations which we see as *doing mathematics* in this context.

[7]More recently, we have chosen to do mathematics problems (which do not necessarily arise out of problems of practice) for the purpose of exploring the relationship between tasks and discourse within our own learning community. We are currently developing a paper out of this aspect of our work.

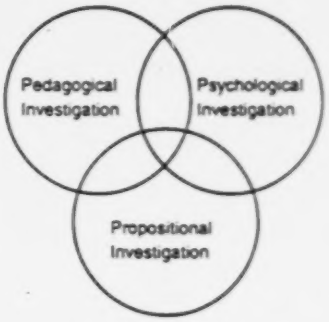

Figure 1. Doing mathematics in our teacher group involves three types of investigations: psychological, pedagogical, and propositional[8]

We have defined these investigations as follows:

- Participants in a psychological investigation explore another person's reasoning. We devote much of the following text to presenting the group's investigation of student Joseph's[9] solution as an example.

- Participants in a pedagogical investigation explore alternative representations, tasks, and strategies for introducing other people to a mathematical idea or facilitating students' learning. We will present one example below in the IMT members' responses to one teacher's problem of practice (see also Pfeiffer, Featherstone, and Smith 1993).

- Participants in a propositional investigation explore particular mathematics problems, i.e., they raise underlying mathematical ideas to the foreground. For example, we pursued a propositional exploration in the first meeting of our group when Helen asked members to write number sentences that equal ten.

We will clarify our representation and our three definitions by examining one IMT conversation which took place in early February 1994. Because of parent-teacher conferences, several of the teachers were absent. In addition, one of the researchers was

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[8]We recognize this figure is inadequate in that it suggests a static view of mathematical conversations. We view the three types of investigations as continually changing in size, in amount of intersection (including times where one circle may not intersect the others), and in relation to foreground and background—each type of investigation being in the foreground with one or both of the others in the background at some times. Furthermore, participants may be doing different investigations simultaneously: one focusing on a psychological investigation while another focuses on a propositional one. Such differences, along with the interests of individual participants, may lead to rapid shifts in investigative focus. Finally, we must note that each participant may be working on a different, private investigation than that which is in public view.

[9]The names used for students in this paper are pseudonyms.

absent. As a result, four teachers and two researchers were present. In the meeting, the members used their exploration of one fourth-grader's mathematical reasoning to help them understand the mathematics embedded in his solution. They also explored the bigger question (for Jane, a fourth-grade teacher, who introduced the issue) of how to use such student mathematical reasoning to help the rest of the fourth grade expand their mathematical knowledge.

Jane's Problem of Practice

The fourth-graders in Jane's class had been working on two-digit by two-digit multiplication. One of Jane's goals for this unit was to help her students find and use efficient ways to solving multiplication problems, including using the standard algorithm for column multiplication. She had the children working on a variety of story problems such as:

> When we returned from camp, each of the students in our class wrote thank-you notes to the counselors. If we each wrote 14 notes, how many did the class write altogether?
>
> [and]
>
> If each student has 25 pencils in the cup on his or her desk, how many pencils are there on all [26] desks?

At the start of the meeting, Jane was insistent that the IMT group help her think about a pedagogical question with which she was struggling. She explained that in her class, after spending time solving the "problem of the day" in their small groups, each group of students would send a representative to the board to present one way their group solved the problem. When class members used this strategy subsequently, they often referred to it as the method of the person who first demonstrated it. For instance, student Arianna's group demonstrated a solution to the first problem above by using the standard algorithm for two-digit column multiplication. This method was known henceforth as "Arianna's method." Student Karl, the representative from another group, divided one two-digit number in half (dividing 14 by 2) so that he could perform single-digit (7) by two-digit (26) multiplication—a procedure with which the class was already familiar—and then

doubled the product. His and others' methods were also named, referred to by name, and used repeatedly thereafter by various members of the class.

The problem of practice which Jane posed to the IMT members was connected to how the students reacted to the various presentations of methods. In most cases, the student demonstrating a solution was using a "homemade" algorithm; e.g., repeated addition, a combination of repeated addition and grouping, division of multiplicand or multiplier in half so he/she could do single-digit multiplication, and then doubling the result. In these cases, the other students might ask their classmates what they did or how they did a certain step or why. In the case of Arianna's method (the standard algorithm), however, students asked no questions—either at the first or at subsequent demonstrations—nor did others appropriate this method. Jane wondered why the students were not questioning Arianna and what she might try in order to encourage them to do so.

Joseph's Solution

As IMT members discussed these questions, Jane decided to show them an "interesting solution" that one of her students had presented. Apparently, she did not view this as more than a brief diversion from her problem of practice, saying, "There is a lot I could focus on, but I want to tell you what Joseph did today." She then went to our white board and copied Joseph's solution to the "pencil problem" (If each student has 25 pencils in the cup on his or her desk, how many pencils are there on all [26] desks?) (Figure 2) from her teaching journal.[10]

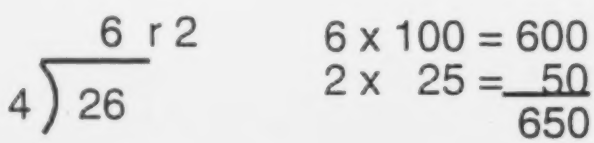

Figure 2. Joseph's solution to the pencil problem

[10]Several of the IMT teachers keep a written record of what students say and write during mathematics discussions.

We invite readers to follow the dialogue and pursue the investigations along with us, so we do not present a step-by-step analysis of Joseph's solution at this point. However, the following picture of Jane's classroom may help readers make sense of the mathematics.[11]

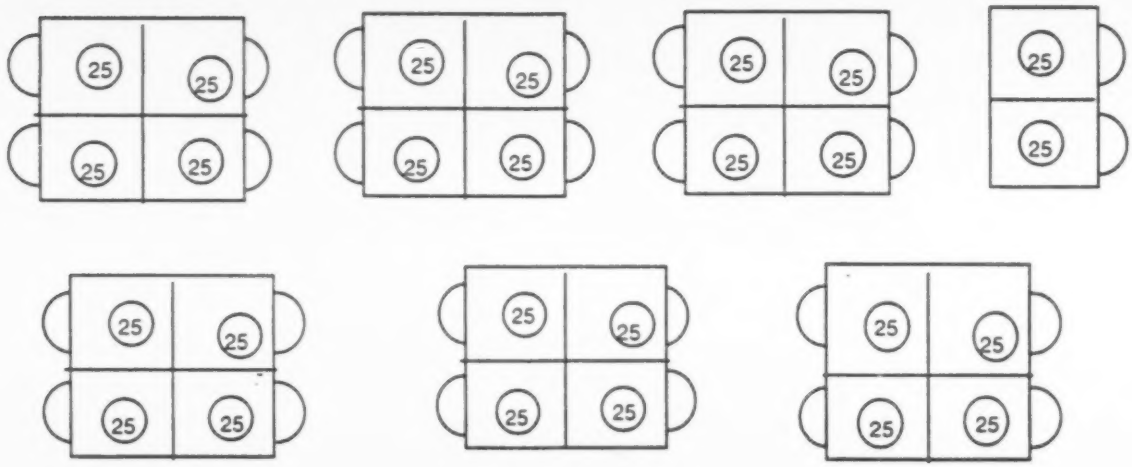

Figure 3. Seating arrangement of Jane's class. The circles of 25s represent the pencils on their desks.

Because the desks in Jane's classroom were pushed into "tables" of four, we can see that each table has 100 pencils.

A Psychological Investigation

The first reaction was Lauren's (Lauren is a researcher/former fourth-grade teacher): Seeing that Jane was about to return to her seat, Lauren asked, "Wait a minute, could you walk us through?" In response, Jane, referring to what she had written on the white board, briefly explained what she thought Joseph's reasoning was: "He said, 'There are four 25s in a hundred, so if I divide my 26 by 4s I know I've got six 100s plus two of that [25].'" Several members commented on the difficulty they were having understanding what Joseph did, and Jane repeated the same explanation. As we pick up

[11]The fact that Jane's classroom was set up in this way may have influenced the way that Joseph thought about the problem. However, the members of the IMT group did not have this picture as they worked to understand his solution.

the conversation, Jane is starting a third, this time different, explanation of Joseph's reasoning.

J: [Jane] He's saying, "I'm not going to multiply this number [26] by 25; I'm going to divide this number [26] by 4 [Kathy, a third-grade teacher: "That's incredible."] and multiply by 100 because essentially I'm going to multiply this by 100."

Teachers usually expect students to multiply 25 x 26 using column multiplication as Arianna did. Joseph, who seems to have been playing with the problem, apparently noticed that 4 x 25 = 100 and believed 100 to be an easy number to multiply. Thus, he seems to have reasoned that by determining how many groups of four 25s there were he could determine how many sets of 100 pencils there were. To put it another way, Joseph might have figured that each group of four students had 100 pencils; if he figured out how many groups of four he could get out of 26 students, he would be able to find the total number of pencils.

L: So, so, I know I can do this. What is it he understands? There's a step he's jumping over, right?

K: Yeah, there's got to be.

L. The longer version of that would be to multiply 25 by 4 and get 100, 26 by 4 and get whatever, 104.

While Jane first described the mechanics *within* Joseph's solution, in this description she seems to be refocusing: on this second try, she described reasoning which enabled Joseph to *reframe* the problem. However, Lauren (L) and Kathy (K) appear to remain focused on his reasoning *within* the problem.

J: No, no. He's saying, "You multiply 25 x 4 and get 100, then I'm going to divide this number [26] *by* 4, then I know how many 100s I need."

K/L: [overlapping] There's an assumption there.

J: [continuing] "And then I got two and a half, and," he said, "there's two more 25s I've got to put in."

K: But how does he know that dividing 26 by 4 will tell how many 100s there are in 26?

J: Because there are four 25s in 100.

We feel the need for a subtle but important distinction here. Joseph did not "multiply 25 x 4." Rather, he may have been assuming the "fact" that 25 x 4 is 100 and not viewing it as performing an operation. In fact, there are lots of assumptions built into this "proof" just as there are in any proof. We conjecture that some of the assumptions Joseph did not make explicit and which appear to have inhibited members' understanding were those Joseph made *before* performing any of the operations which Jane showed the IMT teachers: For example, that you can factor either multiplier or multiplicand (or both) and then multiply the resulting factors in any order.

D: All right, but why?

K: But . .

L: I know, Kathy, I don't [pause] there's an assumption in there.

K: But what I don't get . .

J: Yeah, there is an assumption.

L: It feels intuitive to me and he, he knows . .

K: He knows there's six 100s in 26?

Debi (D), a second-/third-grade coteacher, does not seem to be questioning the truth of Jane's statement but to be asking what role the idea played in Joseph's solution to the problem. Kathy and Lauren (L) seem focused on steps they viewed as missing "in the middle" of Joseph's work. In fact, he apparently reframed the problem from 26 x 25 to m x 4 x 25 (where m is the number Joseph was looking for when he divided 26 by 4) *before* he did any written work.

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Lauren responded (above) to Jane's explanation with a statement followed by two questions. Her statement described her own feelings in relation to this mathematics. She followed this with a question about Joseph's reasoning. In it, she embedded a perception of a connection between Joseph's reasoning and the mathematics of the problem: How does Joseph make connections among the mathematical facts that he knows? Her second question moved her into the mathematics itself; she moved from a psychological focus to a propositional one.

After encouragement from Kathy, Lauren continued the inquiry by trying to reconstruct a "missing" logical step that she believed that Joseph had left unstated. In doing so, she pushed the group into the mathematics. Clearly, she did not yet grasp what Joseph had done: She did not see the significance of the relationship among 4, 25, and 100 which Joseph had made as a crucial first step in his construction of a solution. She knew that 4 x 25 = 100 but did not yet see the role that this equation played in the solution. Lauren's statement that 4 x 26 = 104 showed Jane that Lauren needed help, which Jane tried to provide by saying that 26 ÷ 4 told Joseph how many 100s he needed. Both Kathy and Lauren continued to struggle with the *intermediate* step they believed they needed to identify in order to understand Joseph's solution: They struggled to grasp the relationship between multiplying 25 by 4 and dividing 26 by 4. At this step, their language was focused on Joseph's reasoning rather than on the mathematics. Throughout this portion of the conversation, they focused their attention on either Joseph's reasoning or the mathematics of his solution. They alternately centered their investigations in the psychological and the propositional realms: As one became foreground, the other moved to the background.

We have chosen to view Joseph's solution to the "pencil problem" as having four parts:

1. Decision to reframe the problem (from 26 x 25 or a x b to m x n x b)
2. Construction of and decision to make use of the relationship 4 x 25 = 100
3. Division of 26 by 4
4. Combination of the results of parts 2 and 3 by multiplication (6 x 100 and 2 x 25) followed by addition (600 plus 50)

At the end of this segment of the conversation, Kathy was getting close to piecing the first two parts together, asking, "He knows there's six 100s in 26?" Yet, because she did not yet understand Joseph's reframing of the problem as $\underline{m} \times \underline{4} \times \underline{25}$, she was still bewildered by his solution.[12]

During all of these exchanges, Jane, as Joseph's teacher, acted as the expert on what Joseph did and why. Debi, Kathy, and Lauren directed their questions to her rather than to Steve (researcher/mathematics instructor), even though everyone present saw Steve as the group's "mathematics expert." The teachers might reasonably be expected to direct their questions to Jane since they focused on Joseph's thinking rather than the specifics of the mathematics.[13] One consequence of this focus on Joseph's thinking may have been that members were freer to express their puzzlement because being puzzled by the thinking of a fourth-grader may be more acceptable than being puzzled by fourth-grade mathematics.

K:	He knows there's six 100s in 26?	Debi (D), Kathy (K), and Lauren (L) seem clear that Joseph was making use of a certain relationship among 4, 25, and 100: If one puts four of the 25s together, one gets 100. However they do not appear to see the significance of this relationship for the rest of the solution that Joseph had crafted. They appealed for another explanation (at least Jane took it as such an appeal) of Joseph's reasoning.
L:	(to S) Don't just watch, help.	
K:	Now wait, is that what he knows, Jane?	
J:	He knows that 4 times this number [25] is 100.	
K:	What?	
L:	That much I know.	
K:	That I understand.	
D:	I know that, but what does he do with this?	
J:	I think he's saying, "I don't want to multiply 26; I don't want twenty-six 25s because I'm going to say four of those 25s makes 100."	Jane appears to have recognized that the difficulty members were having understanding Joseph's solution lay in their failure to recognize his reconstruction of the problem.

[12]In fact, for Joseph, these may not have been separable or may have taken place in the reverse order we have listed. It is possible that he saw the relationship of $\underline{4} \times \underline{25} = \underline{100}$ by looking at the "table" where he sat, decided to use the relationship, and then figured out how to do so. However, in the group's exploration, what we have listed as the first part seemed to hold an important key for understanding Joseph's solution.

[13]Their reluctance to ask for Steve's help may have grown out of their experiences of teaching mathematics in new ways: They try to foster classroom norms in which children construct their own understanding of the mathematics at hand through reasoned argument rather than through appeal to experts.

K:	Uh, huh.	The members seem to continue trying to understand the mathematics through an inquiry into Joseph's reasoning. As Jane explained Joseph's reasoning again, she introduced the word "groups" for the first time in connection with 4 x 25. She also commented about the psychology of the other members of her fourth-grade classroom.[14]
J:	"So, I'm really going to multiply a smaller number times 25. [several okays] So, I'm going to divide this [26] by 4. That way I know how many groups of 4 I have to multiply." That's what he's saying. [pause] Now the kids, once he went through it, the kids, they just sort of believed him and let him alone. [overlapping voices] I don't blame them. I get lost.	
J:	What he's saying is, I think this is what's going on in his head.	
K:	Can you draw me a picture?	Kathy, Lauren, and Debi continue to struggle with the mathematics, and Kathy again draws on her classroom experiences.
L:	Just do that again.	
K:	I can't get what he's doing.	

In the above segment, Lauren appealed to the "math expert": "Don't just watch, help." This, however, was not the route Kathy wanted to travel. She kept the conversation focused on the "psychological expert," Jane, by asking, "Now wait, is that what he knows, Jane?" Kathy has had considerable success in developing her understanding of mathematical content through her inquiry into her students' reasoning in mathematical situations (see Featherstone et al. 1995). Perhaps for that reason Kathy adopted a strategy she and her students have constructed together: Rather than turn to a mathematical authority—for example, a textbook or a teacher—for an explanation, they investigate one another's reasoning to develop their own mathematical understanding.

Kathy asked for a new representation to explain the mathematics in Joseph's solution. We conjectured, and Kathy later confirmed, that she was drawing on her experience as a teacher and as a learner in her own classroom. She has learned in the

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[14]Jane's description of what happened in her classroom appears to refer back to an exchange that occurred earlier in the evening when she first raised her pedagogical question, "Why does no one question Arianna about her method?":

C: They ask questions on [methods where] they were familiar with the procedure, like the repeated addition.

S: They could sort of get what the child was doing, and so they could get a handle into . .

K: They could ask questions on . .

D: If you don't understand what they're doing, you can't ask questions.

classroom that pictures help her students to make sense of one another's reasoning and that they also help her to see what children are meaning. Lauren overrode Kathy's suggestion by asking for a repetition of the most recent explanation. Lauren later described herself as feeling that she almost understood Joseph's method; she feared that if another idea or approach was introduced, she would lose what she finally—but tenuously—understood. Thus, she tried to buy time for her understanding to stabilize by asking that the same explanation be repeated.

J: All right. What he's saying . .
S: The first thing that he's doing is saying that 25 x 26 is the same as 26 x 25. Actually, he's saying the other way around.
K: All right, write that down. That's what she's got up there!

Steve, the mathematics "expert," tried to introduce an exploration of one of the mathematical concepts underlying Joseph's solution—the idea that multiplication is commutative. Whereas Lauren, Kathy, and Debi had kept the discussion focused on Joseph's thinking with an intended end of understanding the mathematics, Steve tried to change the focus of the conversation from Joseph's reasoning to the underlying subject matter.

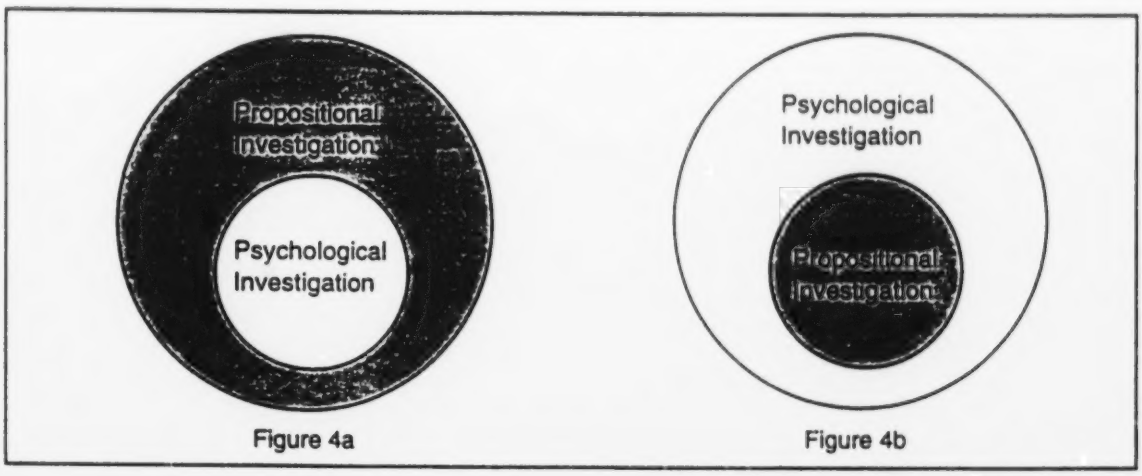


Figure 4a Figure 4b

Figure 4a. The teachers had primarily been investigating Joseph's reasoning against the background of the mathematics

Figure 4b. Steve attempted to reverse the relative positions of the ideas being investigated, moving the propositional into the foreground.

The contrast between Steve's move and the teachers' focus suggests a conjecture: Perhaps members of the group try to center the investigation in a place where they feel comfortable and competent and then foray into the unknown from there. Thus, Steve, who has a graduate degree in mathematics, focused on the mathematics, intending to move from there into Joseph's thinking; other members felt most confident in the area of children's thinking and concentrated their investigations in that domain.

J: What he's saying: "It takes four 25s for a 100."
K: That part I get, but I can't get the next part.
J: Okay. So: "I'm not going to multiply 26 x 25 anymore; I'm going to multiply 26 times the number of groups [pause] here which is 4."
K: But I've got . .
L: Why are you?
K: I don't know . .
J: I don't know. I got lost.
L: You said it.
J: I understood it. I said it a minute ago, and now I can't say it again.
S: Well . .

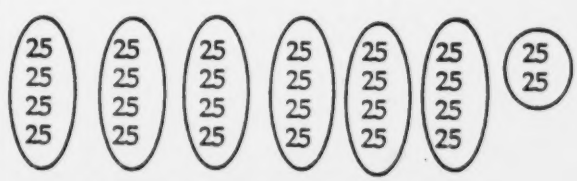
Kathy (K) affirmed she understood the relationship among 4, 25, and 100, but she did not yet see how that relationship connected to the next part of Joseph's solution. Jane (J) apparently becomes confused, and Kathy and Lauren (L) seem unable to help her.

D: He decided to lump four 25s together.
J: Right.
D: So then he [K: "Oh!"] divided the 4 into [K: "Ahh!"] the 26 and got six *groups of 100.*

Debi (D), making use of the idea of groups that Jane had recently introduced, starts an attempt to help Jane by talking about "lumping" the 25s together in groups of fours. It may be that this new language—"lumping four 25s together"—gave Kathy what she needed to make the next step. For, as Debi continued, Kathy seems to "get it."

K: [overlapping] *So, he's got 25, 25, 25, 25.*
J: "But there's two left over so that means there's two more 25s I have to take care of."
K: Yep.
C: [Carole, a second-grade teacher] I still don't understand it.

Kathy followed through on her earlier request for a picture by providing one herself. She moved to the board and started writing 25s, circling groups of four.



Having finished writing twenty-six 25s and circling six groups of four simultaneously with Jane's summation, Kathy appears to have "gotten it."

Over the course of the previous few segments, some of the IMT members pieced together the significance of the relationship that Joseph had constructed among 4, 25, and 100 for his innovative and playful solution.

Debi continued her explanation, making explicit for the first time in the evening that it was *the same 4* in both parts: Joseph grouped the 25s in *4s* and then "divided *the 4* into the 26." As she said this, Kathy used her alternate strategy—drawing a picture. She drew 25s on the board in groups of 4, reaching the end as Jane talked about the two leftover 25s. While Kathy and Debi indicate they had now made sense of Joseph's reasoning, Carole, reflecting on her own psychology, lamented continuing confusion.

In part, this illustrates the elusive nature of understanding. Jane, who had several times confidently explained Joseph's thinking, unexpectedly lost the thread. As the conversation played out though, Debi caught the thread; she picked up and transformed Jane's idea of groups, making new sense of Joseph's reasoning. Conversely, while the last line of this piece of the transcript suggests that Carole did not yet understand Joseph's solution, less than two minutes later, she showed some grasp of it.

Kathy started to explain for Carole, saying, "This is what he does." Her tone of voice indicated Kathy had decided she had a good grasp on Joseph's method and that she felt she could help Carole understand it. As she proceeded through her explanation, Carole completed it, indicating that she had in fact made sense of the method. At the same time, Lauren was clearly still putting the pieces together; she continued to wonder, "But why is he doing it?" In her question, Lauren attempted, through Joseph's thinking, to discover the underlying mathematics: Why does the method work? What mathematical relationships are involved?—for they had all agreed that it does in fact get the right answer.

In response, Jane redescribed what Joseph had done in a more colloquial way. Several members, now that they felt they understood the solution and the mathematics involved, expressed admiration for Joseph's mathematical reasoning. Teachers raised questions about how he was able to construct this solution. They wondered what he had written in his journal, apparently seeking more insight into his reasoning as they prepared to push into pedagogical issues.

Steve led a foray into the underlying structural aspects of the mathematics Joseph used in this solution. This turned into a comparison of Joseph's method and Karl's method which was repeated halving of one factor until he reached a one-digit number (which would not work in this instance since half of 26 is an *odd, two-digit* number). While Steve argued they were the same, Lauren felt there were differences.[15] Looking back, we agree with Lauren. The reasoning behind the two approaches is different in an important way. Karl appears to have managed the complexity of two-digit by two-digit multiplication by constructing an algorithm. He approached such problems by:

1. Halving one factor (for example, in multiplying 14 x 15, he would halve 14 to get 7)
2. Multiplying a two-digit number by a one-digit number (he would multiply 7 x 15, giving 105)
3. Doubling of the product from step 2 (doubling 105 gives 210)

Joseph redefined the pencil problem in a different way. He apparently saw a relationship he wanted to use (4 x 25 = 100) and divided 26 by 4 so he could make use of that relationship. In contrast with Karl, Joseph appears to manage the problem by drawing on what he knows about our number system and about a number that *is not mentioned* in the problem—100. He seems to have noticed the relationship 4 x 25 = 100, held that in mind while he broke down the 26 into 4s, and then combined the results of the two embedded operations. Joseph's method will apply to fewer fourth-grade problems than Karl's: Joseph's method applied to any problem of the form a x 25 while Karl's algorithm works when multiplying 10, 12, 14, 16, 18 (and potentially 24, 28, 32, 36, 40, 48, 56, . . . though we do not have evidence that Karl performed repeated iterations of his method) by any number. Thus, one could say that Karl's solution was more general

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[15]The two methods do share concepts—both are based on the associative property of multiplication—factoring and then multiplying by the factors. In this particular problem, the method presented by Joseph involves the added complexity of the remainder. For a problem such as 24 x 25, the two methods would be nearly identical conceptually. Applying each child's method would involve:

- Factoring one of the original factors (a x b) = (m x n) x b
- Multiplying the other original factor by one of the resultant factors m x (n x b) = m x c
- Multiplying the new product by the other factor m x c = d

than Joseph's. We differentiate the two solutions by describing Joseph's as ad hoc and Karl's as general—though neither is completely so.[16]

In the next few minutes of the IMT meeting, Jane described a second method Joseph had used for solving problems involving two-digit by two-digit multiplication (which Joseph actually demonstrated in her classroom prior to the one described above but which we will call Joseph's "Method 2"): He multiplied each digit separately. For example, in solving 12 x 27, Joseph multiplied 2 x 7 and 2 x 20 and then added the results. He then multiplied 10 x 7 and 10 x 20 and added the results. Finally, he added the subtotals. As a group, IMT members acknowledged this as evidence that Joseph understood the concept of two-digit multiplication well and noted how his method parallels the steps of the traditional algorithm.

Our reflections on this discussion of what constitutes and contributes to understanding in mathematics suggest that, for teachers pursuing the vision of mathematics teaching and learning outlined in the *Standards*, opportunities to explore mathematical content and the nature of mathematics as a subject matter which are *embedded in the practice of teaching* may be fruitful. Such embedding may bear fruit both because of the situated nature of the opportunities and because of the access to the mathematics it offers. Their oblique attack on the mathematics in Joseph's solution gave the members time to move around in the problem. By looking at the mathematics in multiple ways, they had an opportunity to make multiple and varied connections. Such multiplicity of connections deepens everyone's grasp of the mathematics.

A Pedagogical Investigation

We now turn to a later portion of the same conversation in which members focused on Jane's problem of practice, bringing pedagogical investigation into the

[16]Technically, each method of solution applies to the same number of problems (equal to the number of natural numbers), but there are five times as many two-digit by two-digit multiplication problems of a type solvable by Karl's method. Joseph's applies to all problems of the form a x 25, though it is not clear that Joseph would approach all such problems in the same way. Conversely, while Karl's method, *in principle*, could work for all multiplication problems, Karl did not use it unless he could obtain a *whole number* single digit. Thus, for him, it was not completely general.

foreground. Here, the members investigated Jane's pedagogical reasoning—actual and possible—rather than Joseph's mathematical reasoning.

Once the IMT members were comfortable with the mathematics of Joseph's solution, they took up Jane's initial pedagogical questions. Before their foray into Joseph's reasoning, Jane had asked how she could encourage students to question one another more. She told the IMT members that she hoped to help the students find efficient ways to solve two-digit by two-digit multiplication problems.

At this point in the discussion, Jane returned the group to this question by asking why Karl (and, implicitly, the rest of the class) did not question Arianna who was using the traditional algorithm. Jane had said earlier that the students frequently investigated the reasoning of classmates who demonstrated "homemade" solutions to problems. Often, solutions developed in the moment lack generalizability, an important aspect of mathematical thinking. As we have noted, Karl's method failed him in the "pencil problem" because when he halved 26 he got 13. This is where we again pick up the IMT conversation.

J:	And [Karl] said, "But there's got to be a way to do this." and he's looking down the board and he's looking at Arianna's way and he said, "'Cause it's gotta work like that." But he didn't direct his question to Arianna and I said, "Why don't you ask Arianna?" And he said, "No, I'm just going to work on this." So, shall I . .	Jane (J) invited student Karl to pursue an investigation of student Arianna's ideas, but he declined. Since Jane wanted to foster mathematical conversation among students which she believed would help them develop understanding of multiplication, Karl's refusal to accept her invitation presented her with a pedagogical problem.
K:	[interrupting] He should talk to Joseph.	
J:	So, my question is [simultaneous comments covered her question but she persisted] I'm lost on what could I do to move them to asking other questions.	Jane emphasized her concern by repeating her question in the face of Kathy's (K) facetious (on the face of it) response and other distractions.
S:	Is there cross pollination? Does one group see another group's work and adopt that, adopt Karl's method?	
J:	Yeah, there's been quite a few kids adopting other people's methods. That's the thing I'm seeing.	Steve took up her question, starting an investigation of her class's psychology: How did students use one another's ideas?

K:	What would be wrong with getting Joseph to put his method up?—I'm crazy about that method [Here Kathy was referring to Joseph's Method 2.]—and saying to Joseph: "You've got 2 x 7 = 14; where did that come from? What does the 7 stand for; what does the 2 stand for? Now you've got 10 x 7; what does the 10 stand for?" And make him . .	Kathy suggested Jane engage the class in a propositional investigation of Joseph's Method 2 as a solution to Jane's pedagogical question. Carole (C) supported her.
C:	Dig it out.	
K:	Dig it out of him.	
J:	Okay.	

Jane returned the group to her pedagogical questions: How could she encourage her students to ask questions about Arianna's method? How could she interest them in the algorithm? She raised these questions through the story of Karl who knew he had the wrong answer, seemed to realize help could come from Arianna's method, yet actively resisted pursuing it. Kathy responded facetiously by suggesting Karl talk to Joseph. This humorous approach soon resurfaced in a serious form as a pedagogical approach: Have Joseph put his Method 2 up and help the class analyze step by step what he had done. Thus, Kathy suggested both a way into the mathematics of the algorithm for the class and a way for the teacher to play a strong role in modeling how students should question.

Implicit in this segment was a recognition that Joseph's Method 2 is analogous to the standard algorithm. Through "digging" in Joseph's mind, Jane could help her class understand the mathematics of the algorithm, or an equally versatile approach whose meaning may be more accessible to them. In fact, Kathy was arguing implicitly that the class might do exactly what the members had just done in the IMT meeting (albeit with a different problem): Explore the mathematics of two-digit multiplication by exploring Joseph's reasoning.

A few minutes later, they returned to how the fourth-graders were making sense of Joseph's reasoning in his solution of the pencil problem.

J:	Not anyone at his table was doing that, but other kids were chunking the 25s. But, so everyone seems to have strategies to deal with these problems. Okay, I'm not concerned in terms of them having strategies to work with that.	Jane (J) made a move to return the conversation to her pedagogical question; she expressed satisfaction with her students having strategies to solve problems. Kathy (K) supported Jane's inquiry by pushing her with a question that group members often ask one another (and themselves). Kathy's question acknowledged Jane's interest while requesting more background; they could not talk about what a teacher should do without knowing her goals. Kathy's question appears intended to force Jane to examine closely her agenda for her class.
K:	So, what's your purpose?	
J:	My purpose is, really, my question is, how can I get them to ask Arianna? Maybe they're not going to because it's so cut and dried, maybe because the language, or because it doesn't look as interesting, or . .	Jane responded with goals for norms of classroom discourse whereas Kathy appeared to be asking about Jane's mathematical goals. Jane also offered possible reasons for students' behavior.
K:	But what do you want them to . .	
J:	I want them to be anxious to learn that.	
L:	*The algorithm?*	Lauren (L) suggested that Jane may be eager for her students to learn the algorithm, but Jane responded that her purpose was for them to have *access* to the algorithm to meet demands outside of her class.[17] Jane's concern was that the children be able to use efficient methods for solving such problems, which may be the algorithm for some students, and that they possess knowledge of standard procedures.
J:	[overlapping] *To be able to [something]* about the algorithm. More for later purposes than anything else. For when they sit down to do a test. I want them to have that common knowledge. It's common mathematical conversation where these [e.g., Joseph's first method] aren't common mathematical conversations. They're wonderful [but]	
L:	Aren't these the stepping stones of the understanding you want them to have?	Lauren tried to make a connection between Jane's pedagogical goals and the mathematics in Joseph's Method 2.
J:	I want them, and [pause] I think they're get, they're moving, they're feeling comfortable with this, this is no, "Gee, what am I going to do?" They get right at it and . .	Jane explained where she thought her class was mathematically.

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[17]Each year about this time, the members have raised issues related to upcoming standardized testing.

K:	If you want to narrow them down to [the algorithm], I think you have to do it, I mean, to me, the obvious thing to do is to use that [Joseph's Method 2].	Kathy made pedagogical suggestions, supported by Lauren, for how Jane might achieve her mathematical goals.
L:	As the link to the algorithm . .	
K:	As the link to the algorithm. they can draw arrows, they can . .	
J:	Um, hum.	

As they continued to delve into issues embedded in Jane's pedagogical goals, Lauren started to move them toward an exploration of mathematics through this investigation. Kathy and Lauren's contributions worked to confirm for themselves and for the others that the steps in Joseph's Method 2 both paralleled those in the algorithm and were appropriate for the fourth-graders in Jane's class. In these interactions, they moved to overlap their understanding of pedagogy with their understanding of two-digit multiplication, thereby making connections between the algorithm and its component steps. Their earlier exploration of Joseph's reasoning (his "method 1") may have enabled members to develop a new way of thinking about two-digit multiplication; one in which the solver is free to restructure a problem through an unconventional form of factoring (dividing 4 into 26 and getting 6 remainder 2).

Following this, they briefly discussed the history of the multiplication algorithm, questioning whether they could expect 9- and 10-year-olds to construct and incorporate in a few weeks an algorithm that took centuries for the mathematical community to develop.

J:	Yeah, I don't think we use discovery forever.	
L:	No.	
J:	If there's a purpose to having that common understanding of a short process for two-digit multiplication.	
L:	But, I don't know that you're left to discovery if you tap into Joseph as a resource.	Jane (J) was puzzling over what her purpose *should* be. She was wondering whether it is important for students to learn the algorithm.
J:	No, I'm just . .	Lauren (L) turned her attention to how she might teach the algorithm.

K:	That's what I'm saying, It's there.	Kathy (K) then reasserted that Joseph's digit-by-digit method could lay the foundation for the rest of the class to investigate the algorithm. She made the point that the mathematics of his method and that of the algorithm are the same.
L:	Let Joseph do a little talking about how he did his trick. I'm sorry, wrong word.[18] How he got into, how he did that. By him getting to explain it provides some stepping stones.	
D:	For kids to start thinking about it.	Debi (D) and Lauren then took this up, suggesting a pedagogical strategy for Jane to use
L:	Who knows, somebody might say, "Hey, that's like Arianna's." But then they got themselves there rather than . .	

Through investigation into pedagogy, here and in the rest of the conversation, the members developed explicit connections between Joseph's Method 2 and the traditional algorithm. They explored the similarities in the mathematics involved. Further, they briefly traveled into the history of the algorithm. In these instances, they deepened their mathematical knowledge by making new connections and seeing new relationships. Their look into history helped them think about convention and invention as aspects of mathematics—aspects often ignored in elementary classrooms. It also helped them see and make use of the similarity between Joseph's solution and the algorithm.

DISCUSSION

In this paper, we have been examining one way "doing mathematics" occurs in a group in which the agenda grows out of the problems of practice that members bring from their classrooms. In this section, we want to discuss how the examples of pedagogical and psychological investigations we have presented fit within our theoretical frame.

As we talked about and reflected on members' experiences in the IMT group, we constructed the theoretical framework which we believe describes ways that the members of the IMT group "do mathematics" (see Figure 1). We see this framework, which continues to evolve as we explore our data, as composed of three types of investigations

[18]Here the group bursts into laughter because Lauren had referred to a previous conversation in which one teacher disparagingly contrasted the "tricks" that math textbooks present (e.g., "invert and multiply") with her own goals for student learning.

that together constitute the process of "doing mathematics" in our group. We also conjecture that this framework describes the doing of mathematics more generally.

The picture we have drawn to represent the interplay among propositional, psychological, and pedagogical investigations does not capture the dynamic nature of the intellectual work we are trying to describe. In the next sections, we further explore the complexity of what it means to hold a broad conception of *doing mathematics*.

Accessing Resources for Investigation

At different times, people will explore mathematics using any of the three types of investigation as a starting point. Given a choice, we conjecture that most people will begin in the area in which they have the most resources to draw on. In the case we have described here, the teachers and Lauren spent the majority of their time pursuing a psychological investigation; they have most expertise and experience in making sense of children's thinking. In contrast, Steve tried to promote a propositional investigation; he has more experience in the subject matter.

In traditional classrooms, teachers emphasize and model propositional investigations. They often assume that the propositional investigation is the only investigation necessary or available. In such classrooms, teachers generally introduce a new topic with some definitions, theorems, and examples (high school and much of college) or just examples (elementary school). Students then work problems from the text until they "understand." Given these experiences with the subject matter, teachers and students equate understanding with ability to use the appropriate algorithm correctly.

When they are moved by the *Standards* (NCTM 1989, 1991) to make changes in their practice, many teachers begin to invite students to present their solutions to mathematics problems at the blackboard. In response, students explore the psychological and pedagogical as well as the propositional aspects of one another's work. What transpires is very similar to the conversation which the IMT members shared in February 1994.

Different contexts—classrooms of various levels, teacher groups—lead to different relationships among the investigations. For example, in a fourth-grade class, the children are mostly "doing mathematics" through psychological and propositional investigations.

This might also be the case in graduate-level mathematics classes. Conversely, teacher education classes may focus more heavily on pedagogical and psychological investigations.

Integrating Investigations

We believe that in many respects these three different investigations are integrated as a complex process; e.g., we do not pursue a psychological investigation of a student's or teacher's mathematical reasoning without engaging with the underlying aspects of mathematics. For instance, when working on Joseph's solution to 26 x 25, for the most part, we were exploring and learning mathematics. We did this through our investigation of Joseph's reasoning. We were not working on propositional mathematics in a vacuum; we were working with mathematical ideas embedded in a problem the teacher had created as an avenue into two-digit multiplication.

We believe these kinds of investigations take place in *all* mathematics learning contexts. In a fourth-grade class where the teacher is trying to implement her vision of the *Standards*, children present their methods and solutions. The teacher and children then question one another about the work: Why did you do this? With this question, the class members try to elicit the mathematical reasoning which students used to solve the problem. In traditional classrooms, this is also the case, but teachers and students (along with principals, administrators, textbook authors, etc.) seem to assume that the teacher's or textbook author's reasoning is transparent—that students just need to understand the propositions presented. However, when the teacher presents the solution to a problem, the students must analyze her reasons to figure out the mathematics. Thus, in a traditional classroom, the concepts are thought to *be* the mathematics, and learning or accumulating concepts is *doing mathematics*. When the teacher asks "Why?" in such classrooms, she most often seeks and elicits the steps of an algorithm rather than the student's thinking.

When we look at conversations in the classroom of IMT members, we are struck by the parallels between the investigations in which their students are participating and those in which the teachers engage in the group. In both contexts, participants engage in psychological, pedagogical, and propositional investigations. We demonstrate the parallels

between *doing mathematics* in the classroom and in a teacher group by presenting snippets of dialogue from the mathematical learning communities of two IMT members.

WHAT DOES IT MEAN FOR CHILDREN TO "DO MATHEMATICS" WITHIN A LEARNING COMMUNITY?

To *do mathematics*, the class members investigate one another's reasoning in their approach to or solution of a problem or task. Teachers provide opportunities for children to learn how to engage in investigations when they model and support the children's asking questions of each other.

In what follows, sixth-graders participate in an investigation that appears to be both propositional and psychological.[19] The class was trying to determine the relationship between 0 and .515: $0 < .515$, $0 = .515$, or $0 > .515$? One student, in a confusing and tentative way, had claimed that 0 could be both larger and smaller than .515. For him, the relationship was dependent on whether one considered 0 as ".0" or as "0." (decimal point before or after).

Teacher:	Is there somebody who knows what Jorge is getting at? What is Jorge asking about? Emily?
Emily:	That there's, um, that there's a whole number, that this ["0."] would be a whole number and that this [".0"] would be a decimal. But, if we added a zero there ["0.515"], that would be . .
Suzanne:	Or, it would be a whole number. Or, if you like, erase the zero from there ["0.515"], you still have a decimal there and you just erase the zero there ["0."], then neither one has a whole number.

To *do mathematics*, the students investigate one another's logic in using a particular representation or set of representations to articulate an idea or demonstrate a relationship. Teachers provide opportunities for children to pursue investigations when they support students' efforts to offer alternative representations for the ideas they are trying to communicate to each other. For example, we present a third-grade class

[19]The example is from the sixth-grade class of Lisa, one of the IMT teachers.

exploring commutativity.[20] The students have been writing and talking about how many number sentences with two addends they can find for each whole number. Jacques has claimed there is only one number sentence for $\underline{0} + \underline{0} = \underline{0}$ because, "You can't turn [$\underline{0} + \underline{0}$] around." He has also given $\underline{4} + \underline{4} = \underline{8}$ as an example of a case where "you can't turn it around." This transcript begins with Caitlin apparently pursuing a propositional and pedagogical investigation of Jacques' representation. Charles first appears to be shifting to a psychological investigation when he offers to restate what Jacques is explaining. However, he then demonstrates his understanding with a new representation, suggesting that his intent may have been pedagogical. Jacques and Donald pursue Charles's point with further pedagogical exploration.

C:	I know why you can't turn it around. Because if you turned it around, then you have to put another zero here and a zero there [pointing].	Caitlin (C) presents the results of her own propositional investigation in the form of a pedagogical claim.
J:	I know, but you can't turn it around because it is the same number, and it will look the same. So, so, how can you turn it around?	Jacques (J) seems to agree while arguing a different rationale.
C:	You can't.	
J.	You can't.	
H:	I know what he means.	Charles (H) appears to assert that he has performed a psychological investigation.
K:	[Teacher Kathy] Charles, you want to come up and show us? Maybe Jacques and Caitlin, you sit down and listen.	
H:	He means like . .	
K:	I'm going to move this; that will give you some space.	
H:	He means like somebody don't got anything in their hand and, um, they gave it to you, but they wanted it back; they took that back. It's like you had it first and then they had it.	Based on his psychological investigation, Charles offers a pedagogical experiment: He presents an alternative representation.
J:	No, that's *not* what I mean. What I mean is you can't turn it around at all.	Jacques asserts there was an error in Charles's psychological investigation.
K:	[to Charles] Listen to what he's saying.	The teacher suggests a way Charles might improve his psychological investigation.

[20]The example is from the third-grade class of Kathy, one of the IMT teachers.

J:	You can't turn it around at all. `Cause if you turned it around, it would still be the same thing; 4 would be 4, so you couldn't turn it around. See if it was like that [erases and rewrites 4 + 4 = 8], you still could turn it around. Donald?	Jacques seems to be combining propositional (How many addition number sentences can you write for each number?) and pedagogical (How can you show your idea?) investigations.
D:	[Donald] If you tried to turn this around, you won't know if they're turned around or not. They're the same number. There's no way you can know you've turned it around.	

The students in these classrooms also participate in propositional investigations. Returning to the sixth-grade classroom comparing 0 and .515, we can provide the following example:

Jorge:	What if you put the decimal on the other side of the 0 [".0"]?
Teacher:	Okay, what happens if you put the decimal on the other side of the 0?
Joanna:	It would be zero tenths. So, it's not worth anything.
Suzanne:	Yeah, it's still not worth anything.

This joint work involves learners in investigations that are psychological and pedagogical, as well as propositional. In other words, these elementary school students work to make sense of what their peers say and do (conduct *psychological* investigations), look for ways to represent their own ideas and those of classmates (conduct *pedagogical* investigations), and investigate mathematical ideas (conduct *propositional* investigations).

We argue that pedagogical and psychological investigations have always been an integral part of understanding and using mathematical ideas in school. For the most part, learning mathematics in school has been about children solving problems, getting answers, and showing their work. In doing so, they necessarily engage in psychological investigations of the reasoning embedded in the example, whether presented by teachers or students on the chalkboard or by authors in a textbook. However, since textbooks generally present mathematics as given rather than as human thought or construction, few mathematics educators, teachers, or students appear to have made explicit that this is

what students have been doing. In the examples we have given, students and teachers pay conscious and explicit attention to pedagogical and psychological investigations, in addition to propositional ones.

CONCLUSION

To a great extent, the task the IMT group created for itself in the meeting we have just described met the criteria for a worthwhile mathematical task as the NCTM (1991, p. 25) defines it for students:

- engage [IMT members'] intellect;
- develop [IMT members'] mathematical understandings and skills;
- stimulate [IMT members] to make connections and develop a coherent framework for mathematical ideas;
- call for problem formulation, problem solving, and mathematical reasoning;
- promote communication about mathematics;
- represent mathematics as an ongoing human activity;
- display sensitivity to, and draw on, [IMT members'] diverse background experiences and dispositions;
- promote the development of all [IMT members'] dispositions to do mathematics.

These discussions of two-digit multiplication held the IMT group's collective attention for well over an hour and drew thoughtful contributions from everyone present. No one backed away from the problem of making sense of Joseph's solution despite the fact that understanding came slowly. Over our three-year history, other discussions have been compelling and engaged the group in extensive explorations of issues of teaching. Although we cannot be sure *why* this mathematical discussion proved so compelling for the members of the group, we can make a few observations and raise some questions.

First, the effort to understand Joseph's thinking situates this activity firmly in the ongoing work of teaching. This is what the teachers in the group—and their students—do every day: They struggle to make sense of the thinking of individual children, using what the child writes and says. Quite often this is a difficult task, but the teachers feel entirely committed to it. Indeed, the difficulty of making sense of children's thinking is, for the

members of the IMT group, one of the most compelling reasons for deepening their own knowledge of mathematics. Did the group members embrace the challenge of making sense of Joseph's thinking as an opportunity to do, with the help and support of colleagues, difficult work that they are daily required to do alone? We see this as an occasion in which the members were extending their knowledge of teaching; they were learning the practice of teaching situated in their own practice of teaching.

Second, the task of understanding Joseph's thinking was embedded in a celebration of that thinking, a celebration that is one of the rewards of a difficult endeavor. The pleasure of discovering more about the unexpected and fascinating ways in which children think about and represent mathematical ideas is a frequent theme in the IMT group. The teachers punctuated their investigation of Joseph's thinking with expressions of awe and pleasure. And when members came to a clear understanding of what Joseph did to solve the problem, they celebrated his thinking even more enthusiastically: "Oh, my!" exclaimed Lauren. "Isn't that an amazing thing for him to do?" responded Jane. "How can he *do* that?" asked Kathy admiringly. King (1992) argues that professional mathematicians "do mathematics" because it gives them aesthetic pleasure. Does delight in the discovery of children's mathematical thinking sustain teachers—and teacher groups—in a parallel way? Does it draw them particularly, compellingly into "doing mathematics"?

Third, the pedagogical question had a champion who kept it on the table. Jane had been working for several weeks on two-digit by two-digit multiplication. She was, she insisted, "lost on what could she do to move her students to asking other questions." She needed help, and she brought her colleagues back to her question many times. Did her visible need and her repeated requests for help play an important role in keeping the group focused on her students' thinking?

The analysis of this conversation, we think, can shed some light on the problem of helping prospective and practicing teachers to deepen their knowledge of subject matter. A frequent response to the observation that teachers need to know more mathematics in order to teach in new ways is to urge that they take more mathematics courses. The courses offered by university mathematics departments, however, define "doing

mathematics" far more narrowly than the NCTM *Standards* documents. The doing of mathematics looks very different in most departments of mathematics than it does in the discussion we have just examined. College mathematics courses rarely require students to make sense of the reasoning of anyone except the professor and the authors of the textbook—and even that sense-making is assumed and not valued as an achievement. Nor do they ask students to represent or communicate a mathematical idea to anyone who does not already understand it. Even if we could persuade, or compel, teachers to take more mathematics courses, the experiences they would have in such courses might not develop the kinds of knowledge of or about mathematics that teacher educators and reformers advocate.

We wonder whether prospective and practicing teachers might not engage both more wholeheartedly and more productively in discussions that were rooted in questions that connected to children's thinking about mathematics and to the pedagogical dilemmas of working teachers. And, if they saw such discussions as "doing mathematics," might they not begin to reconsider and redefine their own relationship to this subject matter? Might they not come to see themselves as competent learners and doers of mathematics?

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